International Journal of Theoretical Physics, Vol. 36, No. 11, 1997

Conservation Laws for Cosmological Perturbations

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Received June 25, 1997

We briefly recall the problem of defining conserved quantities such as energy in general relativity, and the solution given by introducing a symmetric background. We apply the general formalism to perturbed Robertson–Walker spacetimes with de Sitter geometry as background. We relate the obtained conserved quantities to Traschen's integral constraints and mention a few applications in cosmology.

1. INTRODUCTION

Consider Maxwell's equations:

$$D_\nu F^{\mu\nu} = \mu_0 j^\mu \tag{1}$$

where the electromagnetic tensor is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, A_μ being the vector potential, where j^μ is the electromagnetic current, μ_0 is a coupling constant between the current and the field it creates, and D_μ is the covariant derivative associated with the metric $g_{\mu\nu}$ (with determinant g). Since $F_{\mu\nu}$ is antisymmetric, they can be rewritten as

$$\partial_\nu(\sqrt{-g}F^{\mu\nu}) = \mu_0\sqrt{-g}j^\mu \tag{2}$$

They therefore yield a conservation law and hence an integral equation. Indeed we have, applying Gauss' theorem,

$$\partial_\mu(\sqrt{-g}j^\mu) = 0 \Rightarrow \partial_0 \int_V \sqrt{-g}j^0\, dV = -\int_{\partial V} \sqrt{-g}j^i\, dS_i \tag{3}$$

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If the volume V is taken to be the whole space and if there are no currents on the boundary ∂V, then we have that the total charge e defined as

$$e \equiv \frac{1}{\mu_0} \int_V \sqrt{-g} j^0 \, dV \tag{4}$$

is constant: $\partial_0 e = 0$. Using Maxwell's equations (2), we can moreover express it as a surface integral:

$$e = \frac{1}{\mu_0} \int_{\partial V} \sqrt{-g} F^{0i} \, dS_i \tag{5}$$

On another hand one can construct a tensor, the stress-energy tensor:

$$-\mu_0 T_{\mu\nu} \equiv F_{\mu\rho} F^{\rho}_{\nu} + \frac{1}{4} g_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \tag{6}$$

which, thanks to Maxwell's equations again, is such that

$$D_\mu T^{\mu\nu} + F^{\nu}_{\mu} j^\mu = 0 \tag{7}$$

Outside the charges and in Minkowski space-time in Cartesian coordinates, this equation is also a conservation law and yields another integral equation:

$$\partial_\mu T^{\mu\nu} = 0 \Rightarrow \partial_0 \int_V T^{0\nu} \, dV = -\int_{\partial V} T^{i\nu} \, dS_i \tag{8}$$

If the field decreases fast enough at infinity, the Cartesian vector P^ν defined by

$$P^\nu \equiv \int_V T^{0\nu} \, dV \tag{9}$$

is therefore constant: $\partial_0 P^\nu = 0$.

These conservation laws and integral equations (3) and (8) are mere consequences of Maxwell's equations. In other words, given charges in arbitrary motion, the field $F_{\mu\nu}$ they create is such that the total charge defined by (4) and, Cartesian coordinates being used, the energy-momentum vector defined by (9) are constant if the boundary terms are zero. In addition, the charge is also given by the surface integral (5).

Now it is well known that those laws and equations reflect in fact the symmetries of the theory. Indeed the conservation of the charge, equation (3a), follows from the requirement that, like Maxwell's equations, the action they derive from

$$S \equiv \int \sqrt{-g} L \, d^4x \qquad \text{with} \quad L = \sqrt{-g} \left(j^\mu A_\mu - \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} \right) \tag{10}$$

be the same (up to a total derivative) in a gauge transformation: $A_\mu \rightarrow A_\mu - \partial_\mu f$.

As for the conservation of the stress-energy tensor (8a), it reflects the homogeneity and isotropy of Minkowski spacetime. Indeed, in Cartesian coordinates the action (10) does not depend explicitly on the coordinates x^μ; thus computing $\partial_\mu L$ using Maxwell's equations then yields a tensor which, after symmetrization, is nothing but (6) (in Cartesian coordinates) and is conserved (Noether's theorem).

This (symmetric) stress-energy tensor $T_{\mu\nu}$ can also be defined as the functional derivative of the action with respect to the metric, and equation (7) follows from the fact that the action is a scalar. Minkowski space-time being maximally symmetric, it possesses 10 Killing vector ξ_μ, such that $D_\mu \xi_\nu + D_\nu \xi_\mu = 0$. Hence outside the charges, equation (7) yields

$$\partial_\mu(\sqrt{-g}T^\mu_\nu \xi^\nu) = 0 \tag{11a}$$

which is the generalization to any coordinate system of the conservation law (8a). The integral equation which ensues is

$$\partial_0 P(\xi) = -\int_{\partial V} \sqrt{-g}T^{i\nu}\xi_\nu \, dS_i \qquad \text{where} \quad P(\xi) \equiv \int_V \sqrt{-g}T^0_\nu \xi^\nu \, dV \tag{11b}$$

In Cartesian coordinates and for the four ξ_ν corresponding to time or space translations, $P(\xi)$ is the vector defined by (9) whose constancy (for an isolated system) therefore reflects the symmetries of Minkowski spacetime.

In general relativity, first, gauge invariance and invariance under coordinate transformations are one and the same thing, so that the notions of "charge" and "energy-momentum" of the gravitational system coalesce.

Second, Einstein's equations

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{12}$$

where $G_{\mu\nu}$ is Einstein's tensor, $\kappa = 8\pi G/c^4$ Einstein's coupling constant, and $T_{\mu\nu}$ the stress-energy tensor of matter (defined as the functional derivative of the matter action with respect to the metric), imply, via the Bianchi identity,

$$D_\mu T^{\mu\nu} = 0 \tag{13}$$

Equations (13) [which are the gravitational analogue of equation (7)] can be manipulated in various ways to yield conservation laws [similar to equations (3a) or (8a)]. Landau and Lifschitz (1962) rewrote them as

$$\partial_\mu[(-g)(T^{\mu\nu} + t^{\mu\nu}_{LL})] = 0 \tag{14}$$

where $t^{\mu\nu}_{LL}$ is some expression quadratic in the Christoffel symbols. Hence the quantity

$$P^{\nu}_{LL} = \int_V (-g)(T^{0\nu} + t^{0\nu}_{LL})\, dV \tag{15}$$

is constant if surface terms vanish. It is the gravitational analogue of the charge (4) or the electromagnetic energy-momentum vector (9). And, like the charge [see equation (5)], it can be expressed as a surface integral, in terms of a "superpotential" from which the pseudotensor $t^{0\nu}_{LL}$ derives. However, P^{ν}_{LL} is not a vector under general coordinate transformations and moreover has the wrong tensorial weight. Hence it does not really qualify as a proper definition of energy-momentum.

Einstein on the other hand applied in 1915 Noether's theorem to the Hilbert action from which his equations derive. Indeed, as a functional of the metric it does not depend explicitly on the coordinates. He therefore also obtained conservation laws:

$$\partial_{\mu}[\sqrt{-g}(T^{\mu}_{\nu} + t^{\mu}_{E\nu})] = 0 \tag{16}$$

where $t^{\mu}_{E\nu}$ is yet another expression quadratic in the Christoffel symbols. Hence the quantity

$$P^{\nu}_{E} = \int_V \sqrt{-g}(T^{0\nu} + t^{0\nu}_{E})\, dV \tag{17}$$

is also constant if surface terms vanish, and can also be written as a surface integral. Despite the fact that it is obtained from Noether's theorem and therefore reflects some properties of spacetime and has the right tensorial weight, (17) does not qualify either as a proper definition of energy-momentum, as it is not a vector under general coordinate transformations. Moreover, $t^{\mu\nu}_{E} \equiv g^{\mu\rho}t^{\nu}_{E\rho}$ is not symmetric and cannot define an angular momentum [this is in fact this problem which led Landau and Lifschitz to (14)–(15)].

As advocated by many authors (for reviews see Katz, 1996; Katz *et al.*, 1996) a possible way out of this problem of defining energy and momentum (and angular momentum) in general relativity is to introduce a background spacetime. We now briefly summarize this approach, following Deruelle *et al.* (1997).

2. DEFINING ENERGY, ETC., WITH RESPECT TO A BACKGROUND

Consider a spacetime $(\mathcal{M}, g_{\mu\nu}(x^{\lambda}))$, a background $(\overline{\mathcal{M}}, \overline{g}_{\mu\nu}(x^{\lambda}))$, and a mapping between these two spacetimes.

Take as Lagrangian density for gravity

$$\hat{\mathcal{L}}_G = \frac{1}{2\kappa}\,[\hat{g}^{\mu\nu}(\Delta^{\rho}_{\mu\nu}\Delta^{\sigma}_{\rho\sigma} - \Delta^{\rho}_{\mu\nu}\Delta^{\sigma}_{\rho\nu}) - (\hat{g}^{\mu\nu} - \hat{\bar{g}}^{\mu\nu})\,\bar{R}_{\mu\nu}] \tag{18}$$

where we have introduced the difference $\Delta^{\lambda}_{\mu\nu}$ between Christoffel symbols in $\mathcal{M}$ and $\overline{\mathcal{M}}$ and where $\bar{R}^{\mu\nu}$ is the Ricci tensor of the background. A caret denotes multiplication by $\sqrt{-g}$. Since the "Δ" are tensors, $\hat{\mathcal{L}}_G$ is a true scalar density.

If we now perform a small displacement $\Delta x^{\mu} = \zeta^{\mu}\Delta\lambda$, where ζ^{μ} is an arbitrary vector field and $\Delta\lambda$ an infinitesimal parameter, and use the fact that $\hat{\mathcal{L}}_G$ is a scalar density, we have that, with L_{ζ} denoting the Lie derivative,

$$\mathrm{L}_{\zeta}\hat{\mathcal{L}}_G - \partial_{\mu}(\hat{\mathcal{L}}_G\zeta^{\mu}) = 0 \tag{19}$$

Computing explicitly $\mathrm{L}_{\zeta}\hat{\mathcal{L}}_G$ from (18), it can be shown (Katz *et al.*, 1996) that there exists an identically conserved vector $\hat{I}^{\mu}$, analogous to the electromagnetic current, such that

$$\partial_{\mu}\hat{I}^{\mu} = 0 \tag{20a}$$

yielding the integral equation

$$\partial_0 P(\zeta) = -\int_{\partial V} \hat{I}^i\, dS_i, \qquad \text{where} \quad P(\zeta) \equiv \int_V \hat{I}^0\, dV \tag{20b}$$

Equations (20a) and (20b) are the gravitational analogue of equations (3) and (11a), (11b).

Now it also follows from (20a) that there exists an antisymmetric tensor $\hat{J}^{[\mu\nu]}$ such that

$$\hat{I}^{\mu} = \partial_{\nu}\hat{J}^{[\mu\nu]} \tag{21}$$

This is the gravitational analogue of Maxwell's equation (2). Hence, just like the electric charge [see equation (5)], $P(\zeta)$ can be expressed as a surface integral:

$$P(\zeta) = \int_{\partial V} \hat{J}^{0i}\, dS_i \tag{22}$$

The explicit expressions for $\hat{I}^{\mu}$ and for $\hat{J}^{[\mu\nu]}$ can be found in Katz *et al.* (1996; see also Deruelle *et al.*, 1997).

The equalities (20)–(22) are valid for all $\{g^{\mu\nu}, \bar{g}^{\mu\nu}, \zeta^{\nu}\}$. They become the Noether conservation laws when the vectors ζ^{ν} are Killing vectors of the background. Therefore, in order to obtain the maximum number of Noether conservation laws, one is led to consider a background with maximal symmetry, in which case ten integral equations (one for each Killing vector) can be written. If the Killing vector refers to the time translations in Minkowski spacetime or the quasi-time translations of de Sitter spacetime, then the corresponding quantity $P(\zeta)$ will be called energy. When one uses the three Killing vectors associated with the Lorentz rotations of Minkowski or the quasi-Lorentz rotations of de Sitter spacetimes, $P(\zeta)$ will be the "position of the center of mass," etc. The introduction of a maximally symmetric background thus allows us to define an energy, etc., even if the physical spacetime does not possess symmetries, globally or asymptotically. The justification for such a terminology can be found in, e.g., Katz *et al.* (1996).

3. THE ENERGY OF COSMOLOGICAL PERTURBATIONS WITH RESPECT TO DE SITTER SPACE

We now apply the formalism summarized above to a perturbed Robertson–Walker spacetime with metric

$$ds^2 = dt^2 - a^2(t)(f_{ij} + h_{ij})\, dx^i\, dx^j \tag{23}$$

f_{ij} is the metric of a 3-sphere, plane, or hyperboloid, depending on whether the index $k = (+1, 0, -1)$:

$$f_{ij} = \delta_{ij} + k\,\frac{\delta_{im}\delta_{jn}x^m x^n}{1 - kr^2} \qquad \text{with} \quad r^2 \equiv \delta_{ij}x^i x^j \tag{24}$$

The scale factor $a(t)$ is determined by Friedmann's equation and $h_{ij}(x^\mu)$ is a small perturbation of f_{ij} in a synchronous gauge. The maximally symmetric background is chosen to be de Sitter space with the same spatial topology as the physical perturbed Robertson–Walker spacetime and its metric will be written as

$$d\bar{s}^2 = \Psi(t)^2\, dt^2 - \bar{a}(t)^2 f_{ij}\, dx^i\, dx^j \tag{25}$$

Equation (25) contains a definition of the mapping for each point of the $t =$ const. hypersurface, up to an isometry. The function $\Psi(t)$ defines the mapping of the cosmic times [and the explicit expression for the scale factor $\bar{a}(t)$].

The explicit expressions of the ten de Sitter Killing vectors when the metric is written under the form (25) can be found in, e.g., Deruelle *et al.* (1997).

The zeroth-order conserved quantities $P_{\mathrm{RW}}(\zeta)$ have been defined and studied by Katz *et al.* (1996). Their perturbations at first order were given in Deruelle *et al.* (1997). The final result is $P(\zeta) = P_{\mathrm{RW}}(\zeta) + \delta P(\zeta)$, with

$$\delta P(\zeta) \equiv \int_V \sqrt{-g}\left(\delta T^0_\mu \zeta^\mu + \frac{1}{2}\,\beta\dot{h}\zeta^0\right) dV + \int_{\partial V} \hat{M}^l\, dS_l = \int_{\partial V} (\hat{B}^l + \hat{M}^l)\, dS_l \tag{26}$$

where we have introduced the notations $\kappa\beta \equiv \dot{a}/a - \dot{\bar{a}}/\bar{a}$ and $\bar{h} \equiv -2f^{ij}h_{ij}$, and where the explicit expressions of the ζ-dependent surface terms M^l and B^l can be found in Deruelle *et al.* (1997).

Using the explicit expressions of the de Sitter/Robertson–Walker Killing vectors corresponding to spatial translations, $\zeta^\mu = P^\mu$, the total momentum of the perturbations is thus defined as

$$\delta P_i(P) \equiv a^3 \int_V dV\, \delta T_i^0 + \int_{\partial V} \hat{M}_i^l\, dS_l = \int_{\partial V} (\hat{B}_i^l + \hat{M}_i^l)\, dS_l \tag{27}$$

Hence the total momentum is the sum of a background and mapping-independent volume integral plus a surface term which does depend on the background and the mapping. The same holds for the total angular momentum.

When it comes now to the de Sitter Killing vectors corresponding to quasi-time translations ($\zeta^\mu = T^\mu$) and quasi-Lorentz rotations ($\zeta^\mu = K^\mu$), equations (26) can be written under the form

$$\delta P(T) = \frac{1}{\Psi}\, \delta P_{\mathrm{Tr}}(T) + \int_{\partial V} f(\hat{M}^l + \hat{C}^l)\, dS_l \tag{28}$$

$$\begin{cases} \delta P^i(K) = \dfrac{1}{\Psi}\, \delta P^i_{\mathrm{Tr}}(K) + \displaystyle\int_{\partial V} (\hat{M}^{li} + \hat{D}^{li})\, dS_l & \text{for} \quad k \neq 0 \\ \delta P^i(K) = \dfrac{1}{\Psi}\, \delta P^i_{\mathrm{Tr}}(K) - \dfrac{1}{2H\bar{a}^2}\, \delta P^i(P) + \displaystyle\int_{\partial V} (\hat{M}^{il} + \hat{E}^{il})\, dS_l & \text{for} \quad k = 0 \end{cases} \tag{29}$$

where

$$\delta P_{\mathrm{Tr}}(T) \equiv a^3 \int_V (\delta\rho - H\delta T_l^0 x^l)\, dV = \int_{\partial V} \hat{B}^l(T)\, dS_l \tag{30}$$

$$\begin{cases} \delta P^i_{\mathrm{Tr}}(K) \equiv a^3 \displaystyle\int_V [x^i\delta\rho + H\delta T_l^0(k\delta^{li} - x^l x^i)]\, \frac{dV}{\sqrt{1 - kr^2}} \\ \qquad = \displaystyle\int_{\partial V} \hat{B}^{li}(K)\, dS_l \qquad \text{for} \quad k \neq 0 \\ \delta P^i_{\mathrm{Tr}}(K) \equiv a^3 \displaystyle\int_V \left[x^i\delta\rho + H\delta T_l^0 \left(\frac{1}{2}\, \delta^{li} r^2 - x^l x^i\right)\right] dV \\ \qquad = \displaystyle\int_{\partial V} \hat{B}^{li}(K)\, dS_l \qquad \text{for} \quad k = 0 \end{cases} \tag{31}$$

and where the explicit expressions for the various surface terms can be found in Deruelle *et al.* (1997).

Hence, the energy and motion of the center of mass of the perturbations are also the sum of volume integrals which are, up to the overall function of time Ψ, background and mapping independent, plus surface terms which do depend on the background and the mapping.

Turning to localized perturbations for which all surface integrals vanish, we see from the form (27)–(31) for the conserved quantities that the resulting constraints are background and mapping independent. As shown in Deruelle *et al.* (1997), they are equivalent to Traschen's (1984) constraints, which have been widely used for treating localized perturbations (also called "causal" or "active") (see, e.g., Abbott *et al.*, 1988; Traschen, 1985; Traschen and Eardley, 1986; Veeraraghavan and Stebbins, 1990; Traschen *et al.*, 1986; Uzan and Turok, n.d.).

ACKNOWLEDGMENTS

The original work on which this short review is based was done with Joseph Katz. We thank him for the most enjoyable collaboration. We also thank Mrs. Mady Smet for her warm hospitality in Peyresq.

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